λ	Shift in tenth-metres.	Atmospheres.
4045·98	0·009	6
4045·98	0·020	$11\frac{1}{4}$
4383·72	0·016	$9\frac{3}{4}$
4383·72	0·026	$11\frac{1}{2}$

We find then that the known direct effect of pressure on the radiation or absorption lines is the same, in quality, in water as in air, that is, we get displacements in the *opposite* direction to that we observe the dark lines to occupy in the spectra of Novæ, and we find further that the amount of shift observed in the spectra of new stars differs not only in this respect but also in degree, thus:—

Spark in water.	New stars.
1. Absorption lines least shifted.	Absorption lines most shifted.
2. Radiation lines most shifted.	Radiation lines least shifted.
3. Absorption shift small.	Absorption shift enormous.

It would thus appear that the pairs of bright and dark lines shown in the spectra of new stars do not arise from the cause which produces the appearances presented in the spectrum of the spark in water.

My thanks are due to Mr. C. P. Butler, who obtained and discussed the photographs of the spark spectra, and who, together with Dr. Lockyer, assisted me in the preparation of the paper, and to Mr. F. E. Baxandall, who checked the wave-lengths of the lines discussed and studied the behaviour of the lines representative of the different phenomena.

"The Differential Equations of Fresnel's Polarisation-vector, with an Extension to the Case of Active Media." By JAMES WALKER, M.A. Communicated by Professor CLIFTON, F.R.S. Received February 8,—Read March 6, 1902.

1. In many problems of physical optics it becomes necessary to know the differential equations that the polarisation-vector of a stream of light has to satisfy, and the boundary conditions that subsist at the interface of media possessing different optical properties.

These are usually obtained by formulating some theory respecting the character of the ether in the media and the nature of the vibrations in a train of waves, but there is an obvious advantage in directly basing our investigations, if possible, on the known experimental laws of the propagation of a luminous disturbance.

This has been done by Voigt* in the case of an ordinary isotropic medium by using the principle of interference that lies at the very basis of the science of physical optics, combined with the fact that the propagational speed of light is independent of the direction of the waves; from the equations thus obtained he then forms an expression that may be regarded as representing the energy of the luminous disturbance, and generalising this he deduces by the principle of least action the equations that refer to other classes of homogeneous media.

There is, perhaps, something artifical in this extension of the expression for the energy, and it is therefore better, when this can be done, to apply to each separate case the method employed by Voigt for isotropic media.

2. This plan of procedure presents no difficulty in the case of ordinary crystalline media. According to Fresnel's laws of double refraction, the polarisation-vectors of the waves that can be propagated in any given direction, are parallel to the axes of the central section of a certain ellipsoid—the ellipsoid of polarisation—parallel to the plane of the waves, and the propagational speeds of the corresponding waves are given by the inverse of the lengths of these axes.

If then the equation of the ellipsoid be

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{23}yz + 2a_{13}zx + 2a_{12}xy = 1 \quad \ldots\ldots \quad (1),$$

we obtain by the ordinary methods of determining maxima and minima

$$\left.\begin{aligned} (a_{11} - \omega^2)\,\alpha + a_{12}\beta + a_{13}\gamma &= \mathrm{F}l, \\ a_{12}\alpha + (a_{22} - \omega^2)\,\beta + a_{23}\gamma &= \mathrm{F}m, \\ a_{13}\alpha + a_{23}\beta + (a_{33} - \omega^2)\,\gamma &= \mathrm{F}n, \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots \quad (2),$$

where

$$\mathrm{F} = (a_{11}\alpha + a_{12}\beta + a_{13}\gamma)\,l + (a_{12}\alpha + a_{22}\beta + a_{23}\gamma)\,m + (a_{13}\alpha + a_{23}\beta + a_{33}\gamma)\,n \quad \ldots\ldots\ldots \quad (3),$$

(l, m, n) being the direction-cosines of the normal, ω the propagational speed of the wave, and (α, β, γ) the direction-cosines of its polarisation-vector.

Now if (u, v, w) be the components of the polarisation-vector, the principle of interference is expressed by

$$u = \Sigma\alpha\mathrm{D}, \quad v = \Sigma\beta\mathrm{D}, \quad w = \Sigma\gamma\mathrm{D}, \quad \mathrm{D} = \mathrm{A}\,.\,\mathrm{Exp.}\{\iota\kappa\,(lx + my + nz - \omega t),$$

* 'Kompendium der Theoretischen Physik,' vol. 2, part V, §§ 6, 7.

where $\kappa = 2\pi/\lambda$, λ being the wave-length, and the differential equations of the vector are obtained by eliminating the exponentials and the direction-cosines from these expressions by the aid of equations (2).

This gives at once

$$(\ddot{u}, \ddot{v}, \ddot{w}) = \nabla^2\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}\right)\Omega$$

$$-\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}\cdot\frac{\partial\Omega}{\partial u}+\frac{\partial}{\partial y}\cdot\frac{\partial\Omega}{\partial v}+\frac{\partial}{\partial z}\cdot\frac{\partial\Omega}{\partial w}\right) \quad\text{.........}\quad (4),$$

where

$$2\Omega = a_{11}u^2+a_{22}v^2+a_{33}w^2+2a_{23}vw+2a_{13}wu+2a_{12}uv \quad\text{......}\quad (5).$$

If we introduce a new vector ϖ, the time-gradient of which is defined by

$$(\dot{\varpi}_1, \dot{\varpi}_2, \dot{\varpi}_3)$$

$$= \left(\frac{\partial}{\partial y}\cdot\frac{\partial\Omega}{\partial w}-\frac{\partial}{\partial z}\cdot\frac{\partial\Omega}{\partial v}, \frac{\partial}{\partial z}\cdot\frac{\partial\Omega}{\partial u}-\frac{\partial}{\partial x}\cdot\frac{\partial\Omega}{\partial w}, \frac{\partial}{\partial x}\cdot\frac{\partial\Omega}{\partial v}-\frac{\partial}{\partial y}\cdot\frac{\partial\Omega}{\partial u}\right) \quad\text{...}\quad (6),$$

equation (4) may be written

$$(\dot{u}, \dot{v}, \dot{w}) = -\left(\frac{\partial\varpi_3}{\partial y}-\frac{\partial\varpi_2}{\partial z}, \frac{\partial\varpi_1}{\partial z}-\frac{\partial\varpi_3}{\partial x}, \frac{\partial\varpi_2}{\partial x}-\frac{\partial\varpi_1}{\partial y}\right) \quad\text{......}\quad (7).$$

Thus Fresnel's theory of double refraction leads to the consideration of three vectors—

(1.) The polarisation-vector D with components u, v, w.

(2.) A vector E with components $\partial\Omega/\partial u$, $\partial\Omega/\partial v$, $\partial\Omega/\partial w$.

(3.) A vector ϖ, such that $\dot{\varpi}$ = curl E,

and D and ϖ are connected by the relation, $\dot{\text{D}}$ = − curl ϖ.

Also the vectors D and ϖ are perpendicular to one another and in the plane of the wave, and the vector E is perpendicular to the vector ϖ and in the direction of the normal to the ellipsoid of polarisation at the point in which the polarisation-vector meets it—that is, it is perpendicular to the ray.

We thus see brought out quite clearly the connection between Fresnel's theory and the electromagnetic theory for crystalline media.

The boundary conditions that must be satisfied at the passage between two crystalline media follow at once, if we assume that the transition takes place by a rapid but continuous change of the properties of the one medium into those of the other, and that the above equations hold within the region where this variation occurs. Taking the interface as the plane $x = 0$, we see that these conditions are the continuity of ϖ_2, ϖ_3, $\partial\Omega/\partial v$, $\partial\Omega/\partial w$; and since the curl of a vector has

no divergence anywhere, we may add to these the continuity of u and ϖ_1; but these conditions are clearly not independent of the former.

3. In the case of active crystals we are on less sure ground: we are without the guidance of Fresnel, who only considered the passage of light along the axis of an active uniaxal crystal, and our knowledge of the laws of propagation of light in such media is less definite.

It is, however, established that in any direction within an active uniaxal crystal there are two streams of permanent type that are oppositely polarised with their planes of maximum polarisation parallel and perpendicular respectively to the principal section, and Gouy* has shown that, neglecting small terms of the second order, the existence of these "privileged" streams may be accounted for by a superposition of the effects of ordinary double refraction and of an independent rotary power possessed by the medium.

If this be so, it is easily shown that to the same degree of approximation we have the following extension of Fresnel's theorem respecting the ellipsoid of polarisation:—

In any direction within an active crystalline medium two oppositely polarised streams can be propagated with their planes of maximum polarisation parallel respectively to the axes of the central section of the ellipsoid of polarisation parallel to the plane of the waves; and the propagational speeds of these waves are respectively in excess or defect of the speeds represented by the reciprocal of the length of either of these axes by an amount directly† proportional to the period of the vibrations of the polarisation-vector and to the ratio of the axes of the elliptic vibrations perpendicular and parallel to that axis of the section.

Thus if σ, σ' be the axes of the section, L, L′ the axes of the elliptic vibration parallel respectively to these axes, and ω_1, ω_2 the propagational speeds of the waves,

$$\omega_1 = \sigma^{-1} + \frac{\tau\rho}{4\pi}\frac{L'}{L} = \sigma'^{-1} + \frac{\tau\rho}{4\pi}\frac{L}{L'},$$

$$\omega_2 = \sigma^{-1} - \frac{\tau\rho}{4\pi}\frac{L}{L'} = \sigma'^{-1} - \frac{\tau\rho}{4\pi}\frac{L'}{L},$$

whence, approximately,

$$\sigma^{-2} = \omega_1^2 - \frac{\rho}{\kappa_1}\frac{L'}{L} = \omega_2^2 + \frac{\rho}{\kappa_2}\frac{L}{L'},$$

$$\sigma'^{-2} = \omega_1^2 - \frac{\rho}{\kappa_1}\frac{L}{L'} = \omega_2^2 + \frac{\rho}{\kappa_2}\frac{L'}{L},$$

where $\kappa = 2\pi/\lambda$.

* 'Journ. de Phys.,' (2), vol. 4, p. 142 (1885).

† By taking the excess or defect inversely, instead of directly, proportional to the period, a $(\partial/\partial t)^2$ is introduced in front of the rotary terms in (12).

Now the components of the polarisation-vector of a stream of elliptically polarised light may be represented by the real parts of

$$u = \bar{\alpha}\mathrm{D},\ v = \bar{\beta}\mathrm{D},\ w = \bar{\gamma}\mathrm{D},\ \mathrm{D} = \mathrm{A}\,.\,\exp.\,\{\iota\kappa\,(lx+my+nz-\omega t)\},$$

bars over the letters representing that they are complex, provided that the ratio $\bar{\alpha}:\bar{\beta}:\bar{\gamma}$ is not real, and if we so choose the origin of time that

$$\bar{\alpha}\mathrm{A} = \alpha\mathrm{L}+\iota\alpha'\mathrm{L}',\quad \bar{\beta}\mathrm{A} = \beta\mathrm{L}+\iota\beta'\mathrm{L}',\quad \bar{\gamma}\mathrm{A} = \gamma\mathrm{L}+\iota\gamma'\mathrm{L}',$$

then (α, β, γ), $(\alpha', \beta', \gamma')$ are the direction-cosines of the axes of the ellipse traced by the extremity of the polarisation-vector and L, L′ are the length of the axes in these directions.

Taking again equation (1) to represent the ellipsoid of polarisation, we obtain in place of equations (2) the two sets of equations

$$\left.\begin{aligned}
\left(a_{11}-\omega^2+\frac{\rho}{\kappa}\frac{\mathrm{L}'}{\mathrm{L}}\right)\alpha+a_{12}\beta+a_{13}\gamma &= \mathrm{F}l\\
a_{12}\alpha+\left(a_{22}-\omega^2+\frac{\rho}{\kappa}\frac{\mathrm{L}'}{\mathrm{L}}\right)\beta+a_{23}\gamma &= \mathrm{F}m\\
a_{13}\alpha+a_{23}\beta+\left(a_{33}-\omega^2+\frac{\rho}{\kappa}\frac{\mathrm{L}'}{\mathrm{L}}\right)\gamma &= \mathrm{F}n
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (8),$$

and

$$\left.\begin{aligned}
\left(a_{11}-\omega^2+\frac{\rho}{\kappa}\frac{\mathrm{L}}{\mathrm{L}'}\right)\alpha'+a_{12}\beta'+a_{13}\gamma' &= \mathrm{F}'l\\
a_{12}\alpha'+\left(a_{22}-\omega^2+\frac{\rho}{\kappa}\frac{\mathrm{L}}{\mathrm{L}'}\right)\beta'+a_{23}\gamma' &= \mathrm{F}'m\\
a_{13}\alpha'+a_{23}\beta'+\left(a_{33}-\omega^2+\frac{\rho}{\kappa}\frac{\mathrm{L}}{\mathrm{L}'}\right)\gamma' &= \mathrm{F}'n
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (9),$$

where F is given by (3) and F′ obtained from it by writing α', β', γ' for α, β, γ.

Whence we have

$$(a_{11}-\omega^2)\,\bar{\alpha}+a_{12}\bar{\beta}+a_{13}\bar{\gamma} = \bar{\mathrm{F}}l-(\mathrm{L}'\alpha+\iota\mathrm{L}\alpha')\,\rho/(\kappa\mathrm{A}) \ \ldots \quad (10),$$

and two similar equations, $\bar{\mathrm{F}}$ being obtained from F by writing $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ for α, β, γ.

Now (α, β, γ), $(\alpha', \beta', \gamma')$, (l, m, n) being the direction-cosines of three vectors at right-angles to one another, we have

$$\alpha' = \gamma m-\beta n, \qquad \alpha = -(\gamma' m-\beta' n),$$

and

$$\mathrm{L}'\alpha+\iota\mathrm{L}\alpha' = (\iota\mathrm{L}\gamma-\mathrm{L}'\gamma')\,m-(\iota\mathrm{L}\beta-\mathrm{L}'\beta')\,n = \iota\,(m\bar{\gamma}-n\bar{\beta})\mathrm{A},$$

whence (10) becomes

$$(a_{11}-\omega^2)\,\bar{\alpha}+a_{12}\bar{\beta}+a_{13}\bar{\gamma} = \bar{\mathrm{F}}l-\iota\,(m\bar{\gamma}-n\bar{\beta})\,\rho/\kappa \ \ldots\ldots\ldots \ (11)$$

and two similar equations.

Hence, from the principle of interference expressed by

$$u = \Sigma\bar{\alpha}\mathrm{D}, \quad v = \Sigma\bar{\beta}\mathrm{D}, \quad w = \Sigma\bar{\gamma}\mathrm{D}, \quad \mathrm{D} = \mathrm{A}\exp.\{\iota\kappa(lx+my+nz-\omega t)\},$$

we obtain, as in §2,

$$(\ddot{u}, \ddot{v}, \ddot{w})$$

$$= \nabla^2\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}\right)\Omega - \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}\cdot\frac{\partial\Omega}{\partial u}+\frac{\partial}{\partial y}\cdot\frac{\partial\Omega}{\partial v}+\frac{\partial}{\partial z}\cdot\frac{\partial\Omega}{\partial w}\right)$$

$$-\rho\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}, \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}, \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) \quad \ldots\ldots\ldots \quad (12),$$

which may be written in the form

$$\dot{\mathrm{D}} = -\operatorname{curl}\varpi, \qquad \dot{\varpi} = \operatorname{curl}\mathrm{E}+\rho\mathrm{D},$$

D, E, and Ω having the same significance as in § 2.

The boundary conditions, obtained as in the former case, are the continuity of ϖ_2, ϖ_3, $\partial\Omega/\partial v$, $\partial\Omega/\partial w$, the interface being $x = 0$, together with the continuity of u and $\dot{\varpi}_1 - \rho u$, since within the transition-layer div. D $= 0$, div. $(\dot{\varpi}-\rho\mathrm{D}) = 0$; the two latter conditions are not independent of the previous four, as

$$\dot{u} = -\left(\frac{\partial\varpi_3}{\partial y} - \frac{\partial\varpi_2}{\partial z}\right),$$

$$\dot{\varpi}_1 - \rho u = \frac{\partial}{\partial y}\cdot\frac{\partial\Omega}{\partial w} - \frac{\partial}{\partial z}\cdot\frac{\partial\Omega}{\partial v}$$

4. When we come to the consideration of magnetically active media, our position is still more uncertain, but the following is suggested as an extension of Fresnel's theorem, being a generalisation of results that appear to be established for isotropic media.

In any direction within a magnetically active crystal two oppositely polarised streams can be propagated that have their planes of maximum polarisation parallel respectively to the axes of the central section of the ellipsoid of polarisation parallel to the plane of the waves: and the propagational speeds of these waves are respectively in excess or defect of the speed represented by the reciprocal of the length of either axis of the section by an amount that is inversely proportional to the period of the vibrations and directly proportional to the length of the axis, to the ratio of the axes of the elliptic vibration perpendicular and parallel to the axis, and to the component perpendicular to the section of a vector dependent upon the intensity of the magnetic field.

Thus if σ, σ' be the axes of the section, l, m, n the direction-cosines of its normal, b_1, b_2, b_3 the components of the vector B deter-

mined by the magnetic field, the propagational speeds ω_1, ω_2 of the waves are given by

$$\omega_1 = \sigma^{-1} + \frac{\pi}{\tau}\,\sigma\,(lb_1 + mb_2 + nb_3)\,\frac{\mathrm{L}'}{\mathrm{L}} = \sigma'^{-} + \frac{\pi}{\tau}\,\sigma\,(lb_1 + mb_2 + nb_3)\,\frac{\mathrm{L}}{\mathrm{L}'},$$

$$\omega_2 = \sigma^{-1} - \frac{\pi}{\tau}\,\sigma\,(lb_1 + mb_2 + nb_3)\,\frac{\mathrm{L}}{\mathrm{L}'} = \sigma'^{-1} - \frac{\pi}{\tau}\,\sigma\,(lb_1 + mb_2 + nb_3)\,\frac{\mathrm{L}'}{\mathrm{L}}.$$

Whence, approximately,

$$\sigma^{-2} = \omega_1^2 - \frac{2\pi}{\tau}\,(lb_1 + mb_2 + nb_3)\,\frac{\mathrm{L}'}{\mathrm{L}} = \omega_2^2 + \frac{2\pi}{\tau}\,(lb_1 + mb_2 + nb_3)\,\frac{\mathrm{L}}{\mathrm{L}'},$$

$$\sigma'^{-2} = \omega_1^2 - \frac{2\pi}{\tau}\,(lb_1 + mb_2 + mb_3)\,\frac{\mathrm{L}}{\mathrm{L}'} = \omega_2^2 + \frac{2\pi}{\tau}\,(lb_1 + mb_2 + nb_3)\,\frac{\mathrm{L}'}{\mathrm{L}}.$$

Proceeding as in the last case we have instead of (11),

$$(a_{11} - \omega^2)\,\bar{\alpha} + a_{12}\bar{\beta} + a_{13}\bar{\gamma} = \bar{\mathrm{F}}l - \iota\frac{2\pi}{\tau}\,(lb_1 + mb_2 + nb_3)\,(m\bar{\gamma} - n\bar{\beta}) \qquad \ldots\ldots\ldots \quad (13),$$

and two similar equations; and applying the principle of interference we obtain the equations

$$\begin{aligned}(\ddot{u}, \ddot{v}, \ddot{w}) &= \nabla^2\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}\right)\Omega - \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}\cdot\frac{\partial\Omega}{\partial u} + \frac{\partial}{\partial y}\cdot\frac{\partial\Omega}{\partial v} + \frac{\partial}{\partial z}\cdot\frac{\partial\Omega}{\partial w}\right)\\ &\quad - \left(b_1\frac{\partial}{\partial x} + b_2\frac{\partial}{\partial y} + b_3\frac{\partial}{\partial z}\right)\left(\frac{\partial\dot{w}}{\partial y} - \frac{\partial\dot{v}}{\partial z}, \frac{\partial\dot{u}}{\partial z} - \frac{\partial\dot{w}}{\partial x}, \frac{\partial\dot{v}}{\partial x} - \frac{\partial\dot{u}}{\partial y}\right) \ldots \quad (14),\end{aligned}$$

which may be written in the form

$$\dot{\mathrm{D}} = -\operatorname{curl}\varpi, \qquad \dot{\varpi} = \operatorname{curl}\mathrm{E} + \mathrm{B}\nabla\dot{\mathrm{D}}.$$

The interface being the plane $x = 0$, the boundary conditions are the continuity of ϖ_2, ϖ_3, $\partial\Omega/\partial v + b_1\dot{w}$, $\partial\Omega/\partial w - b_1\dot{v}$, to which we may add the continuity of u and $\varpi_1 - b_2\partial u/\partial y - b_3\partial u/\partial z$, since within the transition-layer div. $\mathrm{D} = 0$, div. $(\varpi - \mathrm{B}\nabla\mathrm{D}) = 0$: the number of independent conditions is, however, only four, as required for the treatment of magneto-optic reflection and the Kerr effect, since

$$\dot{u} = -\left(\frac{\partial\varpi_3}{\partial y} - \frac{\partial\varpi_2}{\partial z}\right)$$

$$\dot{\varpi}_1 - b_2\frac{\partial\dot{u}}{\partial y} - b_3\frac{\partial\dot{u}}{\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial\Omega}{\partial w} - b_1 v\right) - \frac{\partial}{\partial z}\left(\frac{\partial\Omega}{\partial v} + b_1\dot{w}\right).$$